International Journal of Theoretical Physics, Vol. 31, No. 3, 1992

Static Anisotropic Fluid Spheres in General Relativity with Nonuniform Density

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Received June 17, 1991

An Ansatz developed by Maharaj and Maartens is used to obtain solutions of Einstein's field equations for static anisotropic fluid spheres with nonuniform density. These solutions are matched with the Schwarzschild exterior solution.

1. INTRODUCTION

The study of static anisotropic fluid spheres is important for relativistic astrophysics (Bowers and Liang, 1974). Several solutions have been found using various Ansätze (Casenza *et al.*, 1981; Herrera and Ponce de Leon, 1985; Ponce de Leon, 1987*a,b*; Bayin, 1982; Stewart, 1982). Singh and Singh (1985) developed a method to obtain a class of solutions for charged anisotropic fluid spheres. Maharaj and Maartens (1986) developed an Ansatz in which the energy density and radial pressure need to be specified. An alternative approach involves choosing the "degree of anisotropy," i.e., specifying the magnitude of the stress tensor (Casenza *et al.*, 1981). Maharaj and Maartens (1989) found incompressible (constant energy density) solutions. Maharaj and Maartens (1990) used the same Ansatz and obtained a class of static anisotropic spheres with nonuniform energy density. they assumed the energy density in the form used by Durgapal and Bannerji (1983) and Finch and Skea (1989) for isotropic spheres. They matched their solution with the Schwarzschild exterior solution.

In the present work we choose the energy density in the form used by Kuchowicz (1966), Mehra (1966), and Knutsen (1990) for isotropic spheres. The matching and physical properties of these solutions are discussed.

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0020-7748/92/0300-0545$06.50/0

2. FIELD EQUATIONS

We consider the metric for the interior of the static spherically fluid sphere in the form

$$ds^2 = e^\lambda\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 - e^\nu\, dt^2 \tag{2.1}$$

where λ and ν are functions of radial coordinate r alone and $x^i \equiv (r, \theta, \phi, t)$.

The energy-momentum tensor is of the form

$$T^{ij} = \rho u^i u^j + p h^{ij} + \pi^{ij} \tag{2.2}$$

where $u^i = e^{-\nu/2}\delta^i_4$, ρ is the energy density, p is the isotropic (kinetic) pressure, $h^{ij} = g^{ij} + u^i u^j$ is the projection tensor, and π^{ij} is the anisotropic pressure (stress) tensor.

The invariance of u^i and T^{ij} with respect to the Killing vectors of (2.1) implies that the dynamical quantities constructed from them are invariant. This implies that the dynamical quantities assume the form (Maharaj and Maartens, 1986)

$$\rho = \rho(r) \tag{2.3}$$

$$p = p(r) \tag{2.4}$$

$$\pi^{ij} = \sqrt{3}\, s(r)(C^i C^j - \tfrac{1}{3} h^{ij}) \tag{2.5}$$

where $C^i = e^{-\lambda/2}\delta^i_1$ is a unit radial vector orthogonal to u^i and $|s(r)|$ is the magnitude of the stress tensor. For anisotropic fluid spheres, $S \neq 0$. When $S = 0$, $p_r = p_\perp$, we have an isotropic sphere. Using equations (2.1)–(2.5), we find that the Einstein field equations become

$$\frac{1}{r^2} - e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) = \rho \tag{2.6}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = p + \frac{2s}{\sqrt{3}} \tag{2.7}$$

$$e^{-\lambda}\left[\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu' - \lambda'}{2r}\right)\right] = p - \frac{s}{\sqrt{3}} \tag{2.8}$$

where $p + 2s/\sqrt{3} \equiv p_r$ is the radial pressure and $p - s/\sqrt{3} \equiv p_\perp$ is the tangential pressure. The momentum conservation equation can be written in the form

$$(\rho + p_r)\nu' + 2p_r' + \frac{4\sqrt{3}}{r} s = 0 \tag{2.9}$$

Here primes denote differentiation with respect to r.

Using the mass function (Stephani, 1982)

$$m(r) \equiv \tfrac{1}{2} \int_0^r x^2 \rho(x)\, dx \tag{2.10}$$

and the conservation equation (2.9), we find that the field equations (2.6)–(2.8) reduce to

$$e^{-\lambda} = 1 - \frac{2m}{r} \tag{2.11}$$

$$r(r-2m)v' = p_r r^3 + 2m \tag{2.12}$$

$$\left(\frac{2m'}{r^2} + p_r\right) v' + 2p_r' = \frac{4}{r}(p_\perp - p_r) \tag{2.13}$$

Here we have a set of three equations (2.11)–(2.13) in the five variables m (i.e., ρ), p_r, $p_\perp$, v, and λ. For a solution we must specify two physically reasonable functional relations among the five variables. We choose suitable forms of ρ and p_r.

3. SOLUTIONS OF THE FIELD EQUATIONS

Model I

We assume $\rho(r) = \alpha r^n / R^n$, where α, R, and n are constants and $\alpha > 0$, $R > 0$.

For $n = 0$, we have uniform-density spheres. The mass function becomes

$$m(r) = \frac{\alpha r^{n+3}}{2(n+3)R^n} \tag{3.1}$$

On substitution of this value of $m(r)$, equation (2.11) gives

$$e^{-\lambda} = \left[1 - \frac{\alpha r^{n+2}}{(n+3)R^n}\right] \tag{3.2}$$

Using equation (3.1) in equation (2.12), we get

$$\frac{dv}{dr} = \frac{r p_r + \alpha r^{n+1}/(n+3)R^n}{1 - \alpha r^{n+1}/(n+3)R^n} \tag{3.3}$$

Now by choosing suitable values of p_r we try to obtain the solution of equation (3.3).

We choose

$$p_r = C\left[1-\frac{ar^{n+2}}{(n+3)R^n}\right]^2$$

where C is a constant.

Then from equation (3.3) we have

$$e^{\nu} = K_0\left[1-\frac{ar^{n+2}}{(n+3)R^n}\right]^{-1/(n+2)} \exp\left[\frac{Cr^2}{2}-\frac{aCr^{n+4}}{(n+3)(n+4)R^n}\right] \tag{3.4}$$

where K_0 is an integration constant.

By use of the values of p_r and $m(r)$, equation (2.13) gives

$$p_\perp = \frac{a^2r^{2(n+1)}}{4(n+3)R^{2n}}\left[1-\frac{ar^{n+2}}{(n+3)R^n}\right]^{-1} - \frac{(3n+4)aCr^{n+2}}{4(n+3)R^n}$$
$$\times\left[1-\frac{ar^{n+2}}{(n+3)R^n}\right] + C\left[1-\frac{ar^{n+2}}{(n+3)R^n}\right]^2 + \frac{C^2r^2}{4}$$
$$\times\left[1-\frac{ar^{n+2}}{(n+3)R^n}\right]^3 \tag{3.5}$$

Model II

We assume the energy density to be given by (Mehra, 1966; Knutsen, 1990)

$$\rho = \rho_0\left(1-\frac{r^2}{R^2}\right) \tag{3.6}$$

where ρ_0 is the energy density at the center of a star with radius R.

Then from equations (2.10) and (3.6), we have

$$m(r) = \frac{\rho_0 r^3}{6}\left(1-\frac{3r^2}{5R^2}\right) \tag{3.7}$$

Equation (2.11) leads to

$$e^{-\lambda} = \left[1-\frac{\rho_0 r^2}{3}\left(1-\frac{3r^2}{5R^2}\right)\right] \tag{3.8}$$

with $m(r)$ given by (3.7) and the choice for p_r, if we assume that

$$p_r = C_1\left[1 - \frac{\rho_0 r^2}{3}\left(1 - \frac{3r^2}{5R^2}\right)\right]\left(1 - \frac{r^2}{R^2}\right)^n, \qquad n \geq 1 \tag{3.9}$$

then equation (2.12) gives

$$\begin{aligned} e^{\nu} = {} & K\left[1 - \frac{\rho_0 r^2}{3}\left(1 - \frac{3r^2}{5R^2}\right)\right]^{-1/4} \\ & \times\left[\frac{2\rho_0 r^2}{5R^2} - \frac{\rho_0}{3} - \left(\frac{\rho_0^2}{9} - \frac{4\rho_0}{5R^2}\right)^{1/2}\right]^{(5\rho_0)^{1/2}R/4(5\rho_0 R^2 - 36)^{1/2}} \\ & \times\left[\frac{2\rho_0 r^2}{5R^2} - \frac{\rho_0}{3} + \left(\frac{\rho_0^2}{9} - \frac{4\rho_0}{5R^2}\right)^{1/2}\right]^{(5\rho_0)^{1/2}R/4(5\rho_0 R^2 - 36)^{1/2}} \\ & \times \exp\left[-\frac{CR^2}{2(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right] \end{aligned} \tag{3.10}$$

where K is an integration constant.

Further using the value of p_r and the mass function in equation (2.13), we obtain

$$\begin{aligned} p_{\perp} = {} & \frac{C\rho_0 r^2}{4}\left(1 - \frac{r^2}{R^2}\right)^{n+1} + \left(\frac{7C\rho_0 r^4}{20} - \frac{C\rho_0 r^2}{4}\right)\left(1 - \frac{r^2}{R^2}\right)^n \\ & + \left[C\left(1 - \frac{r^2}{R^2}\right)^n + \frac{C^2 r^2}{4}\left(1 - \frac{r^2}{R^2}\right)^{2n} - \frac{nCr^2}{R^2}\left(1 - \frac{r^2}{R^2}\right)^{n-1}\right] \\ & \times\left[1 - \frac{\rho_0 r^2}{3}\left(1 - \frac{3r^2}{5R^2}\right)\right] + \frac{\rho_0^2 r^2}{12}\left(1 - \frac{r^2}{R^2}\right)\left(1 - \frac{3r^2}{5R^2}\right) \\ & \times\left[1 - \frac{\rho_0 r^2}{3}\left(1 - \frac{3r^2}{5R^2}\right)\right]^{-1} \end{aligned} \tag{3.11}$$

4. PHYSICAL PROPERTIES OF THE SOLUTION

The solutions can be matched at the boundary $r = R$ with the Schwarschild exterior solution

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 - \left(1 - \frac{2M}{r}\right) dt^2 \tag{4.1}$$

Model I

Continuity of g_{11} implies that

$$\frac{2M}{R}=\frac{\alpha}{(n+3)}R^2 \tag{4.2}$$

This is satisfied when $(n+3)>0$.

The continuity of g_{44} implies that

$$\frac{2M}{R}=1-\left\{K_0\exp\left[\frac{CR^2}{2}-\frac{\alpha CR^4}{(n+3)(n+4)}\right]\right\}^{(n+2)/(n+3)} \tag{4.3}$$

Thus, constants α, C, and K_0 depend on the mass M and radius R of the fluid sphere.

At the center of the fluid sphere, we have

$$e^{-\lambda}=1, \qquad \rho=0, \qquad e^{\nu}=K_0, \qquad p_r=p_\perp=C \tag{4.4}$$

Thus, we can choose $K_0=1$.

From equation (4.2), we have

$$R=\left[\frac{2(n+3)M}{\alpha}\right]^{1/3} \tag{4.5}$$

and so C can be computed from equation (4.3) when $R>2M$.

Model II

In the second model, at the center of the fluid sphere, we have

$$\rho=\rho_0, \qquad e^{-\lambda}=1, \qquad e^{\nu}=\text{const}, \qquad p_r=p_\perp=C_1 \tag{4.6}$$

As $p_r>0$, $C_1>0$.

The continuity of g_{11} implies that

$$\frac{2M}{R}=\frac{2\rho_0R^2}{15} \tag{4.7}$$

The continuity of g_{44} implies that

$$1-\frac{2\rho_0R^2}{15}=K\left[\frac{\rho_0}{15}-\left(\frac{\rho_0^2}{9}-\frac{4\rho_0}{5R^2}\right)^{1/2}\right]^{(5\rho_0)^{1/2}R/4(5\rho_0R^2-36)^{1/2}}$$

$$\times\exp\left[-\frac{C_1R^2}{2(n+1)}\right] \tag{4.8}$$

Therefore, the constants can be expressed in terms of M, R, and ρ_0. Also, at the boundary of the sphere $p_r = p_\perp = 0$. Therefore, the fluid sphere has equal radial and tangential pressures at the center $r=0$ (the common value being C_1). The pressures go on decreasing continuously and finally they become zero at the boundary $r=R$ of the sphere. The constants should be chosen to ensure this condition.

The surface redshift $z=(1-2M/R)^{-1/2}-1$ can be calculated easily. In addition, the conditions

$$\frac{dp_r}{d\rho} \le 1, \qquad \frac{dp_\perp}{d\rho} \le 1, \qquad \rho_r < \rho, \qquad 0 < r_\perp < \rho$$

put further conditions on the constants.

ACKNOWLEDGMENTS

G. P. Singh would like to express his sincere thanks to the CSIR, New Delhi, for financial support of this work.

REFERENCES

Bayin, S. S. (1982). *Physical Review D*, **26**, 1262.
Bowers, R. L., and Liang, E. P. T. (1974). *Astrophysical Journal*, **188**, 657.
Casenza, M., Herrera, L., Esculpi, M., and Witten, L. (1981). *Journal of Mathematical Physics*, **22**, 118.
Durgapal, M. C., and Bannerji, R. (1983). *Physical Review D*, **27**, 328.
Finch, M. R., and Skea, E. F. (1989). *Classical and Quantum Gravity*, **6**, 467.
Herrera, L., and Ponce de Leon, J. (1985). *Journal of Mathematical Physics*, **26**, 2302.
Knutsen, H. (1990). *General Relativity and Gravitation*, **22**, 925.
Maharaj, S. D., and Maartens, R. (1986). *Journal of Mathematical Physics*, **27**, 2517.
Maharaj, S. D., and Maartens, R. (1989). *General Relativity and Gravitation*, **21**, 899.
Maharaj, S. D., and Maartens, R. (1990). A class of anisotropic spheres, CNLS, Preprint No. 90-1, Johannesburg, South Africa.
Mehra, A. L. (1966). *Journal of the Australian Mathematical Society*, **6**, 153.
Ponce de Leon, J. (1987*a*). *General Relativity and Gravitation*, **19**, 797.
Ponce de Leon, J. (1987*b*). *Journal of Mathematical Physics*, **28**, 1114.
Singh, T., and Singh, D. K. (1985). *National Academy of Mathematics India*, **3**, 55.
Stephani, H. (1982). *General Relativity*, Cambridge University Press, Cambridge.
Stewart, B. W. (1982). *Journal of Physics A*, **15**, 2419.